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UNIQUE MAXIMUM PROPERTY OF THE

STIRLING NUMBERS OF THE SECOND KIND

by

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ABSTRACT:

Letting $f(n)$ and $\ell(n)$ the first and last maximum of the graph $S(n,k); k = 1, 2, \ldots, n$, Kanold [J. Reine Angew. Math 230(1968), 211-212] shows that, for sufficiently large n, $n/\log n < f(n) \leq \ell(n) \leq n\, h(n)/\log n$ with $h(n)$ subject only to $h(n) \to \infty$ as $n \to \infty$. This result was subsequently improved by Harborth [J. Reine Angew. Math 230(1968), 213-214] to yield $\lim_{n\to\infty} f(n)n^{-1}\log n = \lim_{n\to\infty} \ell(n)n^{-1}\log n = 1$. Together with Harper's result [Ann. Math. Stat. 38(1968), 410-414], it is concluded that $S(n,k)$ have, asymptotically, a single maximum. Lieb [J. of Comb. Theory 5(1968), 203-206] shows that Stirling numbers of the second kind possess the property of Strong Logarithmic Concavity, and thus are unimodal. Dobson [J. of Comb. Theory 5(1968), 212-214 and Vol. 7(1969), 116-121] shows a similar result in a stronger form. However, the classical problem of establishing that $S(n,k)$ possess a "unique" maximum for all $n \geq 3$ remains unsolved. It is the purpose of this paper to provide the complete solution of this classical problem.

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Letting $f(n)$ and $\ell(n)$ be the first and last maxim~~um~~a of the graph $S(n,k); k = 1, 2, \ldots, n$, Kanold [J. Reine Angew. Math 230(1968), 211-212] shows that, for sufficiently large n, $n/\log n < f(n) \le \ell(n) \le n\,h(n)/\log n$ with $h(n)$ subject only to $h(n) \to \infty$ as $n \to \infty$. This result was subsequently improved by Harborth [J. Reine Angew. Math 230(1968), 213-214] to yield $\lim_{n\to\infty} f(n)n^{-1}\log n = \lim_{n\to\infty} \ell(n)n^{-1}\log n = 1$. Together with Harper's result [Ann.

Math. Stat. 38 (1967), 410-414], it is concluded that S(n,k) have, asymptotically, a single maximum. Lieb [J. of Comb. Theory 5 (]968), 203-206] shows that Stirling numbers of the second kind possess the property of Strong Logarithmic Concavity, and thus are unimodal. Dobson [J. of Comb. Theory 5 (1968), 212-214 and 7 (1969), 116-121] shows a similar result in a stronger form. However, the classical problem of establishing that S(n,k) possesses a "unique" maximum for all $n \geq 3$ remains unsolved. It is the purpose of this paper to provide the complete solution of this classical problem.

I. Introduction

The Stirling numbers of the second kind $S(n,k)$ have come into renewed salience, primarily due to the fact that $S(n,k)$ is the number of partitions of an n-set into k disjoint nonempty subsets and $S(n,k)$ is the number of distinct fields defined on a finite sample space with n elementary events to which each field contains exactly 2^k events [1]. Letting $f(n)$ and $\ell(n)$ be the first and last maxima of the graph $S(n,k); k = 1, 2, \ldots, n$, Kanold [2] shows that, for sufficiently large n, $n/\log n < f(n) \leq \ell(n) \leq n\, h(n)/\log n$ with $h(n)$ subject only to $h(n) \to \infty$ as $n \to \infty$. This result was subsequently improved to yield

$$\lim_{n\to\infty} f(n)n^{-1} \log n = \lim_{n\to\infty} \ell(n)n^{-1} \log n = 1,$$

by Harborth [3]. Together with Harper's result [4], it is concluded that $S(n,k)$ have, asymptotically, a single maximum. Earlier Lieb [5] shows that Stirling numbers of the second kind possess the property of Strong Logarithmic Concavity, and thus are unimodal. Dobson [6, 7] shows a similar result in a stronger form. However, the classical problem of establishing that $S(n,k)$ possess a~~n~~ "unique" maximum for all $n \geq 3$ remains unsolved. It is the purpose of this paper to provide the complete solution of this classical problem.

II. Unique Maximum Property of $S(n,k)$.

In Riordan [8; p.43] has given the Taylor series

$$\sum_{n=k}^{\infty} S(n,k) z^{n-k} = \prod_{j=1}^{k} (1-jz)^{-1}, \tag{1}$$

convergent for $|z| < k^{-1}$, as a generating function for the Stirling numbers $S(n,k)$ of the second kind. The reciprocal transformation $z = w^{-1}$ converts (1) to the Laurent series

$$\sum_{n=k}^{\infty} S(n,k) w^{-n} = \prod_{j=1}^{k} (w-j)^{-1}, \tag{2}$$

convergent for $|w| > k$. The coefficient in the series (2) may be expressed as the contour integral

$$S(n,k) = \frac{1}{2\pi i} \int_C \frac{w^{n-1} dw}{(w-1)(w-2)\ldots(w-k)} \tag{3}$$

where the contour C encloses the singular points of the integrand. From (3) it follows that

$$S(n,k-1) - S(n,k) = \frac{1}{2\pi i} \int_C \frac{w^{n-1}(w-k-1) dw}{(w-1)(w-2)\ldots(w-k)} . \tag{4}$$

In Milne-Thomson [9;p.11] we find that (4) is the divided difference [123..k] of order k-1 of the polynomial

$$f(w) = w^n - (k+1)\, w^{n-1} . \tag{5}$$

But by [9;p.10] we find that (4) can also be represented by a formula of Hermite as the repeated definite integral

$$(6)\qquad S(n,k-1)-S(n,k) = \int_0^1 dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{k-2}} f^{(k-1)}(u_1)\,dt_{k-1}$$

where $u_1=1+t_1+t_2+..+t_{k-1}$. We imagine that $t_1,t_2,..,t_{k-1}$ constitute a set of rectangular Cartesian coordinates and impose an orthogonal transformation of coordinates to $u_1,u_2,..,u_{k-1}$. We then perform the integration of (6) over the variables $u_2,u_3,..,u_{k-1}$. Because of the structure of the subspace orthogonal to u_1 we find that (6) becomes

$$(7)\qquad \begin{aligned} S(n,k-1)-S(n,k) &= \int_1^k f^{(k-1)}(u_1)g(u_1)\,du_1 \\ &= \int_{1-v}^{v-1} f^{(k-1)}(v+\xi)g(\overset{v+\xi}{\xi})\,d\xi \end{aligned}$$

where $v=(1+k)/2=u_1-\xi$ and g is a positive even function of ξ independent of n. By differentiation of (5) we obtain

$$(8)\qquad f^{(k-1)}(u_1) = (n-1)!\,[nu_1^{\,n-k+1}-(n-k+1)(k+1)u_1^{\,n-k}]/(n-k+1)!.$$

On substituting (8) in (7) we find that the even part of the integrand is proportional to

$$(9)\qquad g(\overset{v+\xi}{\xi})\{n\xi[(v+\xi)^{n-k}-(v-\xi)^{n-k}]-v(n-2k+2)[(v+\xi)^{n-k}+(v-\xi)^{n-k}]\}.$$

Since we are interested in finding more than one pair of n and k values which make (7) vanish, and since $g(\overset{v+\xi}{\xi})$ is independent of n, we see that (9) must be identically zero for all ξ. But (9) vanishes identically only for $n=k=2$.

We have established the following Theorem: The Stirling numbers of the second kind $S(n,k)$ possess a "unique" maximum for $n\geq 3$.

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